TOPOLOGY - III, SOLUTION SHEET 14

Exercise 1. Note that if X is a path-connected space, then for any map $f : X \to X$, the induced map $f_* : H_0(X) \to H_0(X)$ is the identity map. Indeed, $H_0(X)$ is identified with the Abelian Group $\mathbb{Z}$ where the generator 1 can be taken to the class in homology of any point $p \in X$. Therefore f_* sends 1 to 1.

Since $X = D^n$ is contractible, the higher homology groups vanish and we obtain that $\tau(f) = 1$ for any map $f : D^n \to D^n$. The Lefschetz fixed point theorem then implies that f has fixed points.

Exercise 2. Note that translation maps on $\mathbb{R}$ do not have fixed points. However $\mathbb{R}$ is contractible space and just like in exercise 1, one can show that $\tau(f) = 1$ for any map $f : \mathbb{R} \to \mathbb{R}$.

Exercise 3. Since the homology groups of S^n are isomorphic to $\mathbb{Z}$ in degrees 0, n and are equal to 0 in all other degrees, we have that if f is a map from S^n to itself, then $\tau(f) = 1+(-1)^n deg(f)$. Recall that the degree of the anti-podal map $i : S^n \to S^n$ is $(-1)^{n+1}$. So $\tau(f) \neq 0$, whenever $deg(f)$ is not the degree of the anti-podal map, in which case the Lefschetz fixed point theorem implies that f has fixed points.

Exercise 4.
(1) Recall from sheet 13 that $H_k(\mathbb{RP}^n) = \mathbb{Z}$ if $k = 0$ or if $k = n$ with n odd, $H_k(\mathbb{RP}^n) = \mathbb{Z}/2\mathbb{Z}$ if k is odd and not equal to n and $H_k(\mathbb{RP}^n) = 0$ otherwise. By the universal coefficient theorem, we know that $H_*(X;\mathbb{Q}) = H_*(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ and so we make the note that $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$ and $(\mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$. Therefore if n is even, then $\tau(f) = 1$ for any $f : \mathbb{RP}^n \to \mathbb{RP}^n$ and the Lefschetz fixed point implies that f has fixed points.
(2) Note that a linear map $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ induces a continuous map $\overline{f} : \mathbb{RP}^n \to \mathbb{RP}^n$ and that $\overline{f}$ has a fixed point if and only if f has an eigenvector. Indeed if n is odd then the linear map $f(x_1, ..., x_{n+1}) = (x_2, -x_1, x_4, -x_3, ..., x_{n+1}, -x_n)$ has no real eigenvalue.

Exercise 5. We have that $H^*(\mathbb{CP}^n, \mathbb{Q}) = \mathbb{Q}[t]/t^{n+1}$ with t in degree 2. Hence $H^{2k}(\mathbb{CP}^n, \mathbb{Q}) \cong \mathbb{Q}$ for $k = 0, ..., n$ and given a map $f : \mathbb{CP}^n \to \mathbb{CP}^n$, we have that $f^* : H^2(\mathbb{CP}^n, \mathbb{Q}) \to H^2(\mathbb{CP}^n, \mathbb{Q})$ is given by multiplying by a rational number q. Since f is a ring homomorphism it follows that $f^* : H^{2k}(\mathbb{CP}^n, \mathbb{Q}) \to H^{2k}(\mathbb{CP}^n, \mathbb{Q})$ is given by the multiplication map by q^k. Hence $\tau(f) = 1 - q + q^2 - ... + (-1)^n q^n$ which does not have rational solutions if n is even. So the Lefschetz fixed point theorem implies that $f : \mathbb{CP}^n \to \mathbb{CP}^n$ has a fixed point if n is even.

Exercise 6. Recall that the order of a biholomorphic function on a complex domain must be 1. Therefore if f is an automorphism of a Riemann Surface X, with fixed point x_0 then

$\text{index}_{x_0}(f) = 1$. It follows from the Lefschetz-Hopf theorem that $\tau(f)$ counts the number of fixed points of f and is in particular non-negative. Now we compute $\tau(f)$ supposing that f induces the identity map on $H_1(X)$. Clearly f induces the identity map on $H_0(X)$ since X is path-connected and f induces the identity map on $H_2(X) \cong \mathbb{Z}$, since it is orientation preserving. We have that $\tau(f) + 1 - 2g + 1 = 2 - 2g < 0$ since $g > 1$ by assumption. But this is absurd since the number of fixed points is non-negative.